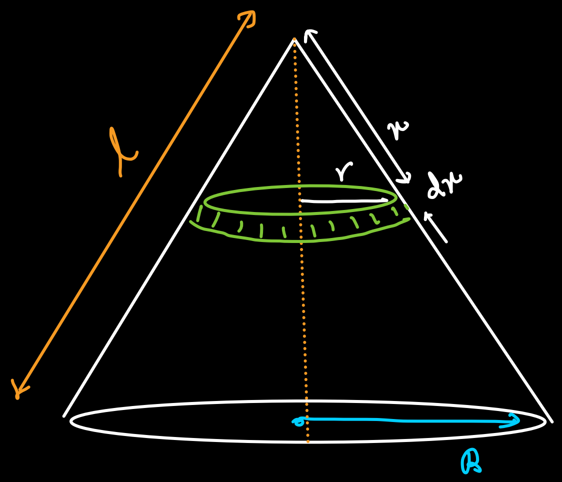
Surface Area of Cone

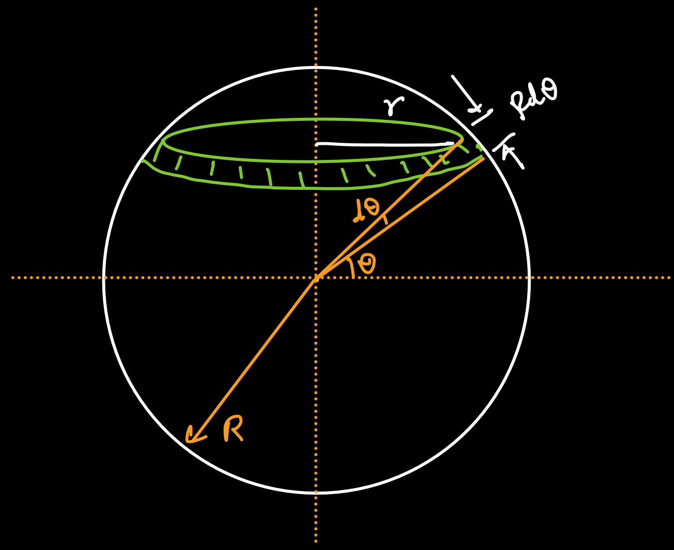


$$dA = 2\pi r \, dx$$

$$= 2\pi\left(\frac{x}{l} R\right) dx$$

$$A = \int_0^l \frac{2\pi R}{l} \, x \, dx$$

Surface area of Sphere

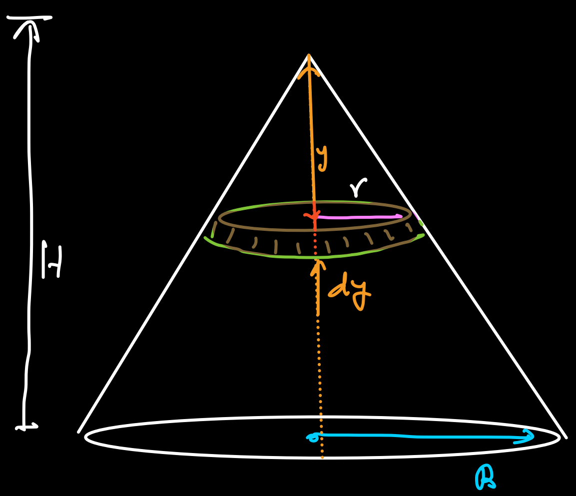


$$dA = 2\pi r \, R d\theta$$

$$= 2\pi (R \cos\theta) \, R d\theta$$

$$A = \int_{-\pi/2}^{\pi/2} 2\pi R^2 \cos\theta \, d\theta$$

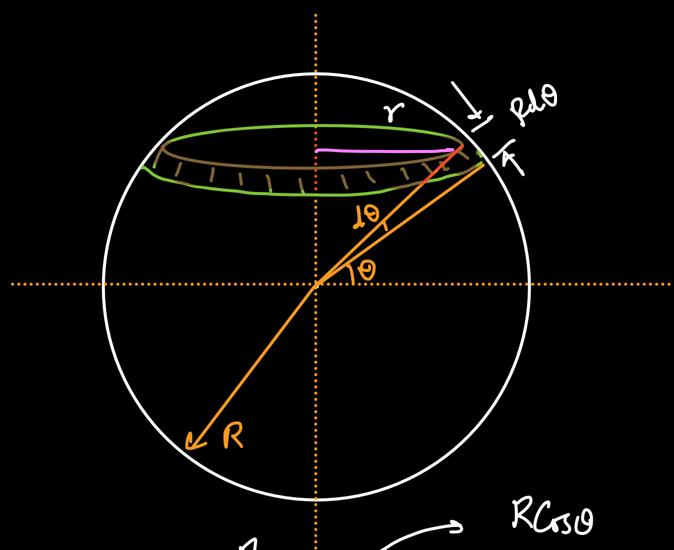
Volume of a Cone



$$V = \int \pi r^2 \, dy$$

$$= \int_0^H \pi \left(\frac{y}{H} \cdot R\right)^2 dy$$

Volume of a Sphere



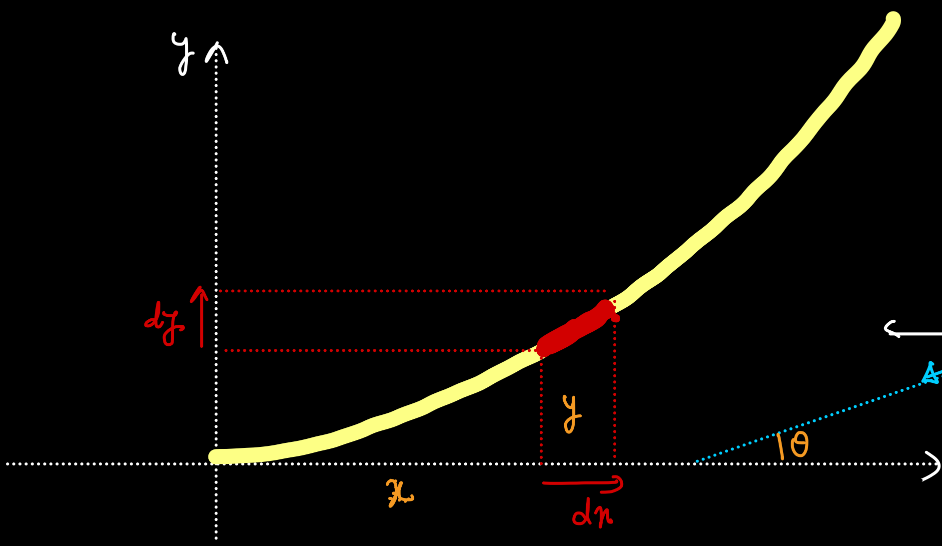
RCosθ

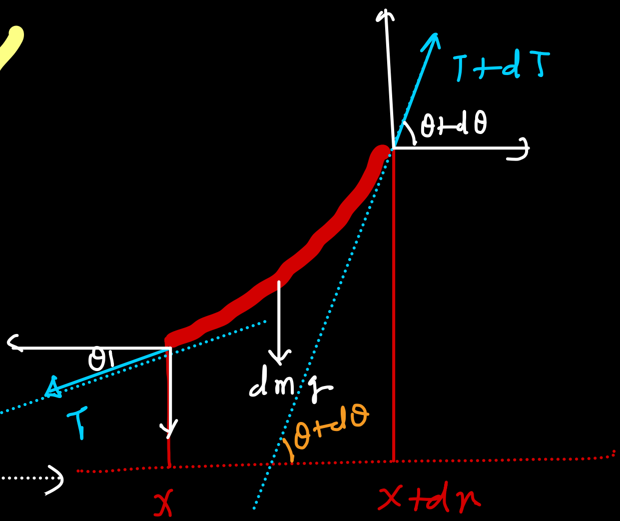
$$V = \int_{-\pi/2}^{\pi/2} \pi r^2 \cdot R d\theta \cos\theta$$

2.8. Hanging chain ****

(a) A chain with uniform mass density per unit length hangs between two given points on two walls. Find the general shape of the chain. Aside from an arbitrary additive constant, the function describing the shape should contain one unknown constant. (The shape of a hanging chain is known as a *catenary*.)

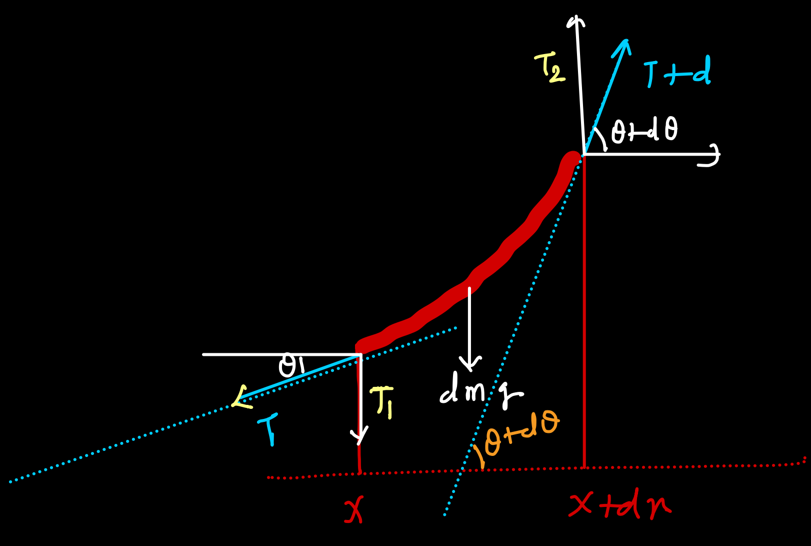
(b) The unknown constant in your answer depends on the horizontal distance d between the walls, the vertical distance λ between the support points, and the length ℓ of the chain (see Fig. 2.14). Find an equation involving these given quantities that determines the unknown constant.





x direction $\quad T\cos\theta = \text{Constant} \qquad T\cos\theta = (T+dT)\cos(\theta+d\theta)$

y direction $\quad (T+dT)\sin(\theta+d\theta) - T\sin\theta = dm\, g$



$$T\cos\theta = K$$

$$\tan\theta = \left.\frac{dy}{dx}\right|_{x} = \frac{T_1}{K}$$

$$\tan(\theta+d\theta) = \left.\frac{dy}{dx}\right|_{x+dx} = \frac{T_2}{K}$$

$$T_2 - T_1 = (dm)\, g$$

λ : mass of unit length

$$K\left[\frac{dy}{dx}\Big|_{x+dx} - \frac{dy}{dx}\Big|_{x} \right] = \lambda \sqrt{(dx)^2 + (dy)^2}\; g$$

$$\frac{K\left[\frac{dy}{dx}\Big|_{x+dx} - \frac{dy}{dx}\Big|_{x} \right]}{dx} = \lambda \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; g$$

$$K \frac{d^2y}{dx^2} = \lambda \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; g$$

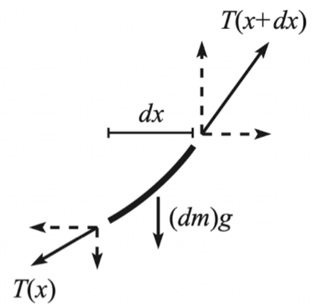


Fig. 2.47

(a) The key fact to note is that the horizontal component, T_x, of the tension is the same throughout the chain. This is true because the net horizontal force on any subpart of the chain must be zero. Label the constant value as $T_x \equiv C$.

Let the shape of the chain be described by the function $y(x)$. Since the tension points along the chain at all points (see Problem 2.12), its components satisfy $T_y/T_x = y'$, which gives $T_y = Cy'$. In other words, T_y is proportional to the slope of the chain.

Now consider a little piece of the chain, with endpoints at x and $x + dx$, as shown in Fig. 2.47. The difference in the T_y values at the endpoints is what balances the weight of the little piece, $(dm)g$. The length of the piece is $ds = dx\sqrt{1+y'^2}$, so if ρ is the density, we have

$$dT_y = (\rho\, ds)g = \rho g\, dx\sqrt{1+y'^2} \quad\Longrightarrow\quad \frac{dT_y}{dx} = \rho g\sqrt{1+y'^2}. \tag{2.31}$$

Using the $T_y = Cy'$ result from above, this becomes $Cy'' = \rho g\sqrt{1+y'^2}$. Letting $z \equiv y'$, we can separate variables and integrate to obtain

$$\int \frac{dz}{\sqrt{1+z^2}} = \int \frac{\rho g\, dx}{C} \quad\Longrightarrow\quad \sinh^{-1} z = \frac{\rho g x}{C} + A, \tag{2.32}$$

where A is a constant of integration. We can make this look a little cleaner if we define constants α and a such that $\alpha \equiv \rho g/C$ and $a \equiv A/\alpha$. We then obtain

$$\sinh^{-1} z = \alpha(x+a) \quad\Longrightarrow\quad z = \sinh\alpha(x+a). \tag{2.33}$$

Recalling that $z \equiv dy/dx$, we can integrate again to obtain

$$y(x) = \frac{1}{\alpha}\cosh\alpha(x+a) + h. \tag{2.34}$$

The shape of the chain is therefore a hyperbolic cosine function. The constant h isn't too important, because it depends simply on where we pick the $y = 0$ height. Furthermore, we can eliminate the need for the constant a if we pick $x = 0$ to be where the lowest point of the chain is (or where it would be, in the case where the slope is always nonzero). In this case, using Eq. (2.34), we see that $y'(0) = 0$ implies $a = 0$, as desired. We then have (ignoring the constant h) the nice simple result,

$$y(x) = \frac{1}{\alpha}\cosh(\alpha x). \tag{2.35}$$

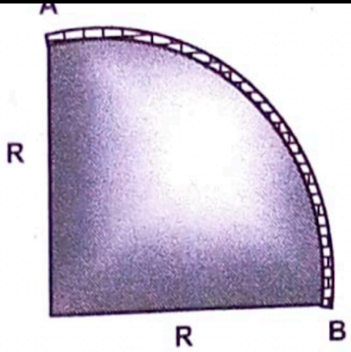
A rope AB of linear mass density λ is placed on a quarter vertical fixed disc of radius R as shown in the figure. The surface between the disc and rope is rough such that the rope is just in equilibrium. Gravitational acceleration is g. Choose the correct option(s).

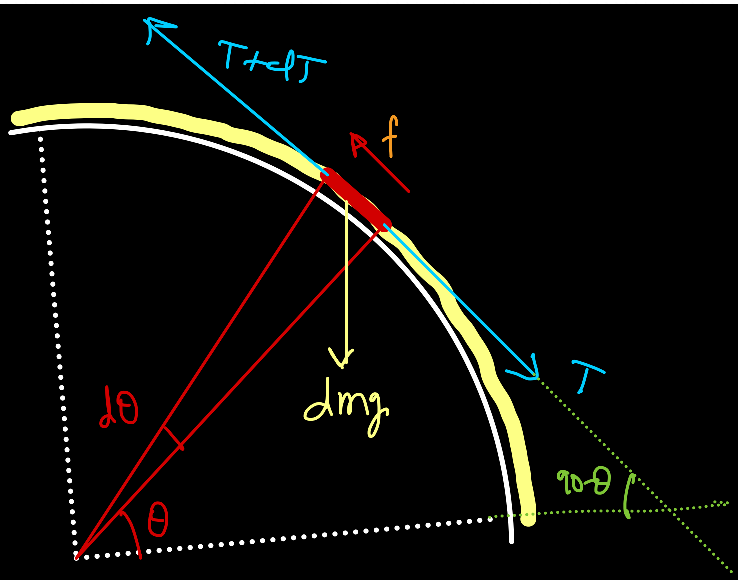
(A) Coefficient of static friction between rope and disc is $\mu = 1$.

(B) Coefficient of static friction between rope and disc is $\mu = \frac{1}{\sqrt{2}}$.

(C) Maximum tension in the rope is at the top most point A of the rope.

(D) Maximum tension in the rope is $\lambda Rg(\sqrt{2}-1)$.





$$dT + f = dmg\cos\theta$$

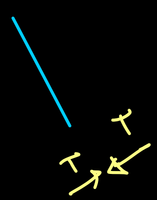
$$dT + \mu\, dmg\sin\theta = dmg\cos\theta$$

$$dT = dmg\cos\theta - \mu\, dmg\sin\theta$$

$$T_1 - T_2 = \int dT = \int_0^{\pi/2} (dmg\cos\theta - \mu\, dmg\sin\theta)$$

$$T_1 - T_2 = \int_0^{\pi/2} ((\lambda R d\theta) g\cos\theta - \mu(\lambda R d\theta) g\sin\theta)$$

$$0 = \lambda Rg(1-\mu) \qquad \mu = 1$$

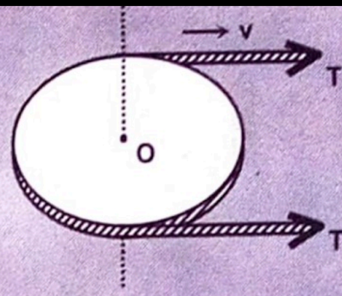


$$dT = dmg\cos\theta - \mu\, dmg\sin\theta$$

$$\frac{dT}{d\theta} = \lambda R g\cos\theta - \mu\lambda R g\sin\theta = 0$$

$$\cot\theta = \mu$$

5. A flexible drive belt runs over a frictionless pulley as shown in figure. The pulley is rotating freely about the vertical axis passing through the centre O of the pulley. The vertical axis is fixed on the horizontal smooth surface. The mass per unit length of the drive belt is 1 kg/m and the tension in the drive belt 8 N. The speed of the drive belt is 2 m/s. Find the net normal force applied by the belt on the pulley in newton.

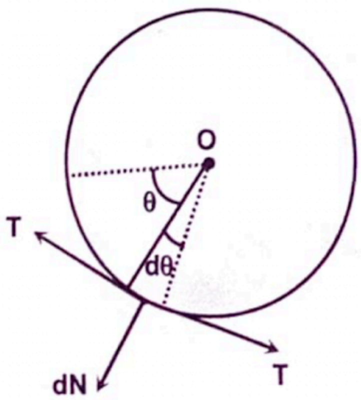


1. $2T\sin\frac{d\theta}{2} - dN = \lambda R d\theta \frac{v^2}{R}$ (λ is linear mass density of belt)

$\therefore\ T d\theta - dN = \lambda v^2 d\theta$

$\therefore$ so total normal $= \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} dN\cos\theta = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} T\cos\theta d\theta - \lambda v^2 \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos\theta d\theta$

$\therefore\ N = 8$ newton



Applications of calculus to kinematics

$$\begin{tikzcd} x \arrow[d, bend right=60, "\frac{dx}{dt}"'] \\ v \arrow[u, bend right=60, "\int v\,dt"'] \arrow[d, bend right=60, "\frac{dv}{dt}"'] \\ a \arrow[u, bend right=60, "\int a\,dt"'] \end{tikzcd}$$

Ex. A particle moves along x axis such that its velocity is given as $V = 4 - t$ m/s. At $t=0$ $x = 2$m. find the distance travelled by the particle in the interval $t=0$ to $t=6$s.

Imp

$$\frac{dx}{dt} = v \qquad\qquad \frac{dv}{dt} = a$$

$$dx = v\,dt \qquad\qquad V_2 - V_1 = \int_{t_1}^{t_2} a\,dt$$

change in velocity

$$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v\,dt$$

$$x\Big|_{x_1}^{x_2} = \int_{t_1}^{t_2} v\,dt$$

displacement $\longrightarrow$ $x_2 - x_1 = \int_{t_1}^{t_2} v\,dt$

method 1

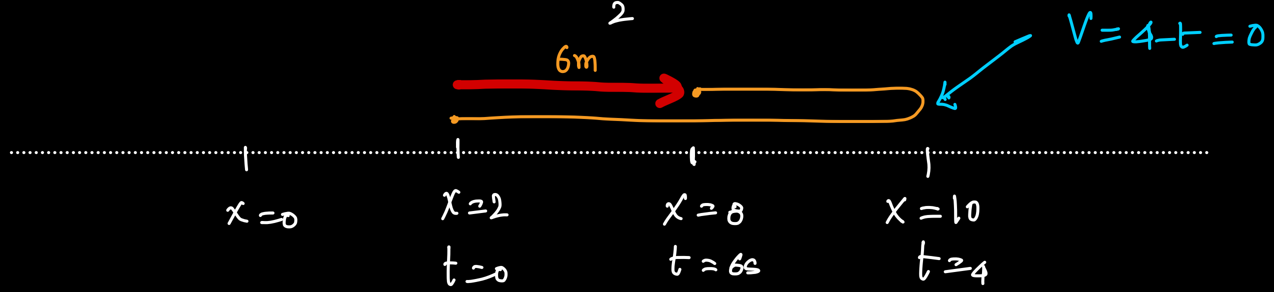
$$x = \int v\,dt$$

$$= \int (4-t)\,dt$$

$$= 4t - \frac{t^2}{2} + c$$

$t = 0 \quad x = 2 \quad c = 2$

$$x = -\frac{t^2}{2} + 4t + 2$$



$$\int_0^6 (4-t)\,dt = 6\text{m}$$ ⟶ only gives displacement

method 2 :

$$\int_0^4 (4-t)\,dt + \left| \int_4^6 (4-t)\,dt \right|$$

General problems for $a \to v \to x$

Ex. For a particle moving along x axis

$a = -3\sin t \qquad t = 0 \quad x = 0 \quad v = 4\,\text{m/s}$

Find position as a function of time ?

method 1

$$a = -3\sin t$$

$$v = \int a\,dt = -3\int \sin t\,dt = 3\cos t + c$$

At $t=0$ $\quad v=4$ $\qquad 4 = 3\cos(0) + c$ $\qquad c = 1$

$$\boxed{v = 3\cos t + 1}$$

$$x = \int v\,dt = \int (3\cos t + 1)\,dt = 3\sin t + t + c$$

At $t=0$ $\quad x=0$ $\qquad c=0$

$$\boxed{x = 3\sin t + t}$$

method 2

$$\frac{dv}{dt} = a = -3\sin t$$

$$dv = -3\sin t\,dt$$

$$\int_4^v dv = \int_0^t -3\sin t\,dt$$

$$v\Big|_4^v = 3\cos t\Big|_0^t$$

$$v - 4 = 3\cos t - 3\cos(0)$$

$$v = 3\cos t + 1$$

$$\frac{dx}{dt} = v = 3\cos t + 1$$

$$\int_0^x dx = \int_0^t (3\cos t + 1)\,dt$$

$$x = 3\sin t + t$$

Ex A particle moves along a straight line. Its acc varies as $a = 4 - v \; m/s^2$

At $t=0$ $x=0$ $v=0$

Sol:

$$\frac{dv}{dt} = 4 - v$$

$$\frac{dv}{4-v} = dt$$

$$\int_0^v \frac{dv}{4-v} = \int_0^t dt$$

$$-\ln(4-v)\Big|_0^v = t\Big|_0^t$$

$$-\left[\ln(4-v) - \ln 4\right] = t - 0$$

$$\ln\left(\frac{4-v}{4}\right) = -t$$

$$\frac{4-v}{4} = e^{-t}$$

$$\boxed{v = 4(1-e^{-t})}$$

$\int \frac{dv}{4-v}$

$z = 4-v$

$\frac{dz}{dv} = -1$

$dz = -dv$

$= -\int \frac{dz}{z}$

$= -\ln z + c$

$= -\ln(4-v) + c$

$\ln A - \ln B = \ln \frac{A}{B}$

$\ln A = B \quad A = e^B$

For $x-t$

$$x = 4(t + e^{-t}) - 4$$